

Title	Spectral properties of massless Dirac operators with real-valued potentials (Spectral and Scattering Theory and Related Topics)
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Citation	数理解析研究所講究録別冊 = RIMS Kôkyûroku Bessatsu (2014), B45: 25-30
Issue Date	2014-04
URL	<a href="http://hdl.handle.net/2433/217419">http://hdl.handle.net/2433/217419</a>
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Type	Departmental Bulletin Paper
Textversion	publisher

# Spectral properties of massless Dirac operators with real-valued potentials

By

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## Abstract

We prove that a Schnol'-type theorem holds for massless Dirac operators under minimal assumptions on the potential, and apply this result to conclude that the spectrum of a certain class of such operators covers the whole real line. We also discuss embedded eigenvalues of massless Dirac operators with suitable scalar potentials.

## § 1. Introduction

This paper is an announcement of results on spectral properties of Dirac operators with real-valued potentials and will be followed by a complete treatment in which all proofs are given.

The Dirac operators to be considered in this paper are

$$(1.1) \quad H_2 = -i \sigma \cdot \nabla + q(x) \quad \text{in } L^2(\mathbb{R}^2; \mathbb{C}^2)$$

and

$$(1.2) \quad H_3 = -i \alpha \cdot \nabla + q(x) \quad \text{in } L^2(\mathbb{R}^3; \mathbb{C}^4).$$

Here  $\sigma = (\sigma_1, \sigma_2)$  and  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$  are given as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad (j \in \{1, 2, 3\}) \quad \text{with} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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Received March 30, 2012. Revised September 11, 2012.

2000 Mathematics Subject Classification(s):

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The dot products are to be read as

$$\sigma \cdot \nabla = \sigma_1 \frac{\partial}{\partial x_1} + \sigma_2 \frac{\partial}{\partial x_2}$$

in (1.1) and

$$\alpha \cdot \nabla = \alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3}$$

in (1.2). The potential  $q$  is a real-valued function on  $\mathbb{R}^d$ , where  $d = 2$  or  $d = 3$ , respectively. The operators  $H_2, H_3$  differ from the standard Dirac operator in that they lack a mass term, usually represented by an additional anti-commuting matrix:  $\sigma_3$  for the two-dimensional case and

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

for the three-dimensional case, where  $I$  is a  $2 \times 2$  identity matrix.

The purpose of the present paper is to show that  $\sigma(H_d) = \mathbb{R}$  under minimal assumptions on  $q$ . In particular, we shall not require any restriction on the growth or decay of the potential  $q$  at infinity.

We have two motivations. First, the spectrum of the one-dimensional massless Dirac operator

$$H_1 = -i\sigma_2 \frac{d}{dx} + q(x) \quad \text{in } L^2(\mathbb{R}; \mathbb{C}^2)$$

covers the whole real axis and is purely absolutely continuous whenever  $q \in L^1_{loc}(\mathbb{R}, \mathbb{R})$ . This surprising fact was first pointed out by one of the authors in [8]. By separation in spherical polar coordinates, this result also implies that  $\sigma(H_d) = \mathbb{R}$  if  $q$  is rotationally symmetric; see [9]. Second, it is believed that the energy spectrum of graphene, in which electron transport is governed by Dirac equations in two dimensions without a mass term, has no bandgap (zero bandgap); see [2], [4], [7]. For these reasons, it is natural to make an attempt to show that  $\sigma(H_d) = \mathbb{R}$  under minimal assumptions on  $q$ .

## § 2. Embedded eigenvalues

It is difficult to imagine that the spectra of  $H_2$  and  $H_3$  are always purely absolutely continuous regardless of  $q$ . Actually, in the three-dimensional case, we have an example of  $q$  which gives rise to a zero mode of  $H_3$ , *i.e.* an example of  $q$  for which  $H_3$  has the embedded eigenvalue 0.

*Example 1.* Let  $q(x) = -3/\langle x \rangle^2$ , where  $\langle x \rangle = \sqrt{1 + |x|^2}$ . Then there exists a unique self-adjoint realization of  $H_3$  in  $L^2(\mathbb{R}^3; \mathbb{C}^4)$  with  $\text{Dom}(H_3) = H^1(\mathbb{R}^3; \mathbb{C}^4)$ , the

Sobolev space of order 1. If one puts

$$f(x) = \langle x \rangle^{-3} (I_4 + i\alpha \cdot x) \phi_0$$

with  $\phi_0$  a unit vector in  $\mathbb{C}^4$ , then a direct calculation shows that  $H_3 f = 0$ . This implies that  $0 \in \sigma_p(H_3)$ , because  $f \in \text{Dom}(H_3)$ . Thus  $H_3$  has a zero mode. Since  $\sigma(H_3) = \mathbb{R}$ , the energy 0 is an embedded eigenvalue of  $H_3$ .

The potential  $q$  and the zero mode  $f$  in Example 1 were motivated by [6].

Example 2 below indicates that there is a difference in spectral property between  $H_3$  and  $H_2$ . In fact, quite a similar construction in Example 2 below gives a zero resonance of  $H_2$ , not a zero mode of  $H_2$ . On the other hand, we do not know if the potential  $q$  in Example 2 gives rise to a zero mode of  $H_2$ .

*Example 2.* Let  $q(x) = -2/\langle x \rangle^2$ . Then there exists a unique self-adjoint realization of  $H_2$  in  $L^2(\mathbb{R}^2; \mathbb{C}^2)$  with  $\text{Dom}(H_2) = H^1(\mathbb{R}^2; \mathbb{C}^2)$ , and  $\sigma(H_2) = \mathbb{R}$ . If

$$\psi(x) = \langle x \rangle^{-2} (I_2 + i\sigma \cdot x) \phi_0,$$

$\phi_0$  a unit vector in  $\mathbb{C}^2$ , then one sees that  $H_2 \psi = 0$ . However, it is clear that  $\psi \notin L^2(\mathbb{R}^2; \mathbb{C}^2)$ . Therefore,  $\psi$  is not a zero mode of  $H_2$ . On the other hand, one finds that  $\psi \in L^{2,-s}(\mathbb{R}^2; \mathbb{C}^2)$  for  $\forall s > 0$ , where

$$L^{2,-s}(\mathbb{R}^2; \mathbb{C}^2) = \{ \varphi \mid \|\langle x \rangle^{-s} \varphi\|_{L^2} < \infty \}.$$

This means that  $\psi$  is a zero resonance of  $H_2$ .

It is not an easy task to clarify whether  $H_d$  has embedded eigenvalues for general potentials. However, we have a good control of the embedded eigenvalues of  $H_d$  if  $q(x)$  is rotationally symmetric. To formulate a result, we need to introduce the definition of the limit range  $\mathcal{R}_\infty(q)$  of  $q$ :

$$\mathcal{R}_\infty(q) = \bigcap_{r>0} \overline{\{q(x) \mid |x| \geq r\}},$$

where  $\overline{A}$  denotes the closure of a subset  $A \subset \mathbb{R}$ .

**Theorem 2.1 (Schmidt[9]).** *Let  $q(x) = \eta(|x|)$  and let  $\eta \in L^1_{loc}(0, \infty)$ . Suppose that there exists a real number  $E \in \mathbb{R} \setminus \mathcal{R}_\infty(q)$  such that*

$$\frac{1}{r(E - \eta(r)) - 1} \in BV(r_0, \infty)$$

*for some  $r_0 > 0$ , where  $BV(r_0, \infty)$  denotes the set of functions of bounded variations on the interval  $(r_0, \infty)$ . Then  $\sigma_p(H_d) \subset \mathcal{R}_\infty(q)$  for  $d \in \{2, 3\}$ .*

Theorem 2.1 is a direct consequence of [9, Corollary 1]. If  $q$  is not rotationally symmetric, we can prove the following.

**Theorem 2.2.** *Let  $q \in C^1(\mathbb{R}^d; \mathbb{R})$ ,  $d \in \{2, 3\}$ , and suppose that both  $q$  and  $(x \cdot \nabla)q$  are bounded functions. Then  $\sigma_p(H_d) \subset [m_q, M_q]$ , where*

$$m_q = \inf_x \{q(x) + (x \cdot \nabla)q(x)\}, \quad M_q = \sup_x \{q(x) + (x \cdot \nabla)q(x)\}.$$

The proof of Theorem 2.2 is based on a virial theorem in an abstract setting; see [1, Lemma 2.1].

### § 3. Schnol's theorem for Dirac operators

We now prepare a Schnol' theorem for Dirac operators. As for Schnol's theorem, we refer the reader [3, p.21, Theorem 2.9] which is a characterization of the spectra of Schrödinger operators in terms of polynomially bounded eigensolutions. In the three-dimensional Dirac operators, the theorem can be stated as follows:

**Theorem 3.1.** *Let  $q \in L^2_{loc}(\mathbb{R}^3; \mathbb{R})$ , and let  $E$  be a real number. Suppose  $f$  is a polynomially bounded measurable function on  $\mathbb{R}^3$ , not identically 0, and satisfies the equation*

$$(3.1) \quad (-i\alpha \cdot \nabla + q)f = Ef$$

*in the distribution sense. Then  $E \in \sigma(H_3)$  for any self-adjoint realization  $H_3$  such that  $\text{Dom}(H_3) \supset H^1(\mathbb{R}^3; \mathbb{C}^4) \cap \text{Dom}(q)$ .*

**Outline of the proof of Theorem 3.1.** We follow the line of the proof of [3, p. 21, Theorem 2.9].

We may suppose  $f \notin L^2(\mathbb{R}^3; \mathbb{C}^4)$  without loss of generality. Let  $\varphi \in C_0^\infty(\mathbb{R}^3)$  such that  $\varphi(x) = 1$  ( $|x| \leq 1$ ) and  $\varphi(x) = 0$  ( $|x| \geq 2$ ), and put  $\varphi_n(x) = \varphi(x/n)$  for  $n \in \mathbb{N}$ . Then introducing a monotonically increasing sequence  $(M(n))_{n \in \mathbb{N}}$  by

$$M(n) = \int_{|x| \leq n} |f(x)|^2 dx,$$

and defining  $f_n := \varphi_n f / \|\varphi_n f\|_{L^2}$ , we can show that there exists a positive constant  $C$  such that

$$(3.2) \quad \|(H_3 - E)f_n\|_{L^2}^2 \leq C \frac{M(2n) - M(n)}{n^2 M(n)}.$$

for all  $n$ .

On the other hand, we can deduce that

$$\liminf_{n \rightarrow \infty} \frac{M(2n) - M(n)}{n^2 M(n)} = 0,$$

which implies that there exists a subsequence  $(M(n_k))_{k \in \mathbb{N}}$  such that

$$(3.3) \quad \lim_{k \rightarrow \infty} \frac{M(2n_k) - M(n_k)}{n_k^2 M(n_k)} = 0.$$

Combining (3.2) with (3.3), we can conclude that  $E \in \sigma(H_3)$ . ■

We give an application of the Schnol' theorem, and show that the spectra of massless Dirac operators with real-valued potentials always coincide with the whole real axis, provided that the potentials are of the form specified in Theorem 3.2 below.

**Theorem 3.2.** *Let  $\eta \in C^1(\mathbb{R}; \mathbb{R})$  and define  $q(x) := \eta(x \cdot k)$  on  $\mathbb{R}^d$ ,  $d \in \{2, 3\}$ , where  $k \in \mathbb{R}^d$  is a unit vector. Then  $\sigma(H_d) = \mathbb{R}$ .*

**Outline of the proof of Theorem 3.2.** We only give the proof for  $d = 3$ . Put

$$\xi(t) = \int_0^t \eta(\tau) d\tau.$$

Choose a unit vector  $\phi_0 \in \mathbb{C}^4$ ,  $\neq 0$ , so that  $(\alpha \cdot k)\phi_0 = \phi_0$ . For a given  $E \in \mathbb{R}$ , define

$$f(x) = e^{-i(\alpha \cdot k)\xi(x \cdot k)} e^{iEx \cdot k} \phi_0.$$

Then  $f$  satisfies the equation (3.1). Moreover,  $f \in C^1(\mathbb{R}^3; \mathbb{C}^4)$ , and  $|f(x)|_{\mathbb{C}^4} = 1$  for all  $x \in \mathbb{R}^3$ . It follows from Theorem 3.1 that  $E \in \sigma(H_3)$ . Since  $E$  is an arbitrary real number, we can conclude that  $\sigma(H_3) = \mathbb{R}$ . ■

#### § 4. The main result

We now state the main theorem, which greatly generalizes Theorem 3.2.

**Theorem 4.1.** *Let  $q \in L^2_{loc}(\mathbb{R}^3; \mathbb{R})$ . Suppose that there is a sequence  $(k_n)_{n \in \mathbb{N}}$  of unit vectors in  $\mathbb{R}^3$ , a sequence  $(B_{r_n}(a_n))_{n \in \mathbb{N}}$  of balls with centre  $a_n \in \mathbb{R}^3$  and radius  $r_n \rightarrow \infty$  ( $n \rightarrow \infty$ ), and a sequence of square-integrable functions  $q_n : (-r_n, r_n) \rightarrow \mathbb{R}$  ( $n \in \mathbb{N}$ ) such that*

$$r_n^{-3} \int_{B(a_n, r_n)} |q(x) - q_n((x - a_n) \cdot k_n)|^2 dx \rightarrow 0$$

as  $n \rightarrow \infty$ . Then  $\sigma(H_3) = \mathbb{R}$  for any self-adjoint extension  $H_3$  of

$$(-i\alpha \cdot \nabla + q)|_{C_0^\infty(\mathbb{R}^3)^4}.$$

The two dimensional analogue of the statements above holds true.

**Outline of the proof of Theorem 4.1.** For a given energy  $E \in \mathbb{R}$ , we shall construct a singular sequence  $(h_n)_{n \in \mathbb{N}}$ , thus showing that  $E \in \sigma(H_3)$ . To this end, we first choose a function  $\tilde{\eta}_n \in C^\infty(-r_n, r_n)$  so that

$$\frac{1}{2r_n} \int_{-r_n}^{r_n} |q_n(t) - \tilde{\eta}_n(t)|^2 dt \rightarrow 0$$

as  $n \rightarrow \infty$ . Then following the idea in the proof of Theorem 3.2, we put

$$\xi_n(t) = \int_0^t \tilde{\eta}_n(\tau) d\tau,$$

and choose a sequence of unit vectors  $\phi_n \in \mathbb{C}^4$ ,  $\neq 0$ , so that  $(\alpha \cdot k_n)\phi_n = \phi_n$ , and define a sequence of functions  $(f_n)_{n \in \mathbb{N}}$  by

$$f_n(x) = e^{-i(\alpha \cdot k_n) \xi_n((x-a_n) \cdot k_n)} e^{iEx \cdot k_n} \phi_n : B_{r_n}(a_n) \rightarrow \mathbb{C}^4.$$

We now choose a function  $\chi \in C_0^\infty(\mathbb{R}^3)$  such that  $\text{supp} \chi \subset B_1(0)$  and that  $\|\chi\|_{L^2} = 1$ . Putting

$$h_n(x) = r_n^{-3/2} \chi\left(\frac{x - a_n}{r_n}\right) f_n(x),$$

we can obtain the desired singular sequence.  $\blacksquare$

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